Coloring sparse random Cayley graphs

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Main result

For an abelian group Z and integer k let (Z, k) be the random Cayley graph formed by choosing $S \subseteq Z$ randomly with $|S| = k$ and connecting $x, y \in Z$ if $x - y \in S \cup (-S)$.

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Theorem (T.)

Let Z be an abelian group of size N. Then

$$\chi\left(Z, \frac{1}{64}\log N\right) \leq 3$$

with high probability $(1 - o_N(1))$.

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with high probability. For any group B and $k \leq N/2$

$$\max\left(\sqrt{\frac{k}{\log N}}, \frac{k^2}{N\log^2 N}\right) \ll \chi(B,k) \ll \frac{k}{\log k}$$

with high probability.

Comparison with Erdős–Rényi

Fact

If $G(N, p)$ is the Erdős–Rényi random graph, then with high probability

$$\chi\left(G\left(N, \frac{\log N}{N}\right)\right) = \omega(1).$$

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If $G(N,p)$ is the Erdős–Rényi random graph, then with high probability

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Proof: Union bound over r-colorings. For r fixed

$$\mathbb{P}\left(G\left(N, \frac{\log N}{N}\right) \text{ r-colorable}\right) \leq r^N \left(1 - \frac{\log N}{N}\right)^{\binom{N/r}{2}} = \exp(-\Omega_r(N \log N)).$$

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For $p > (\log N)^{-1/3+o(1)}$, with high probability

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Less is known outside cyclic groups.

Theorem (Alon–Pham 2025+)

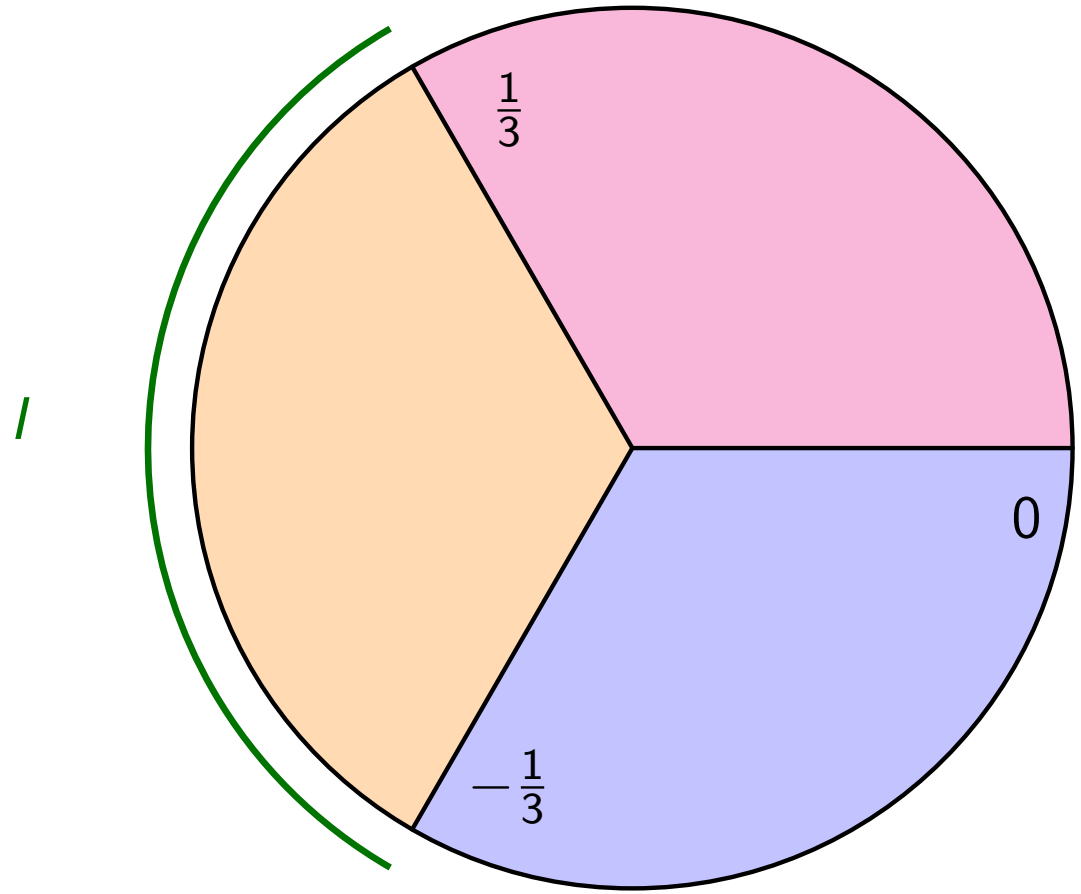
For $p \leq 1/2$, with high probability

$$\alpha(Z, p) = \widetilde{O}\left(p^{-3/2}\right).$$

Coloring mechanism

Partition $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ into thirds and let $I := \mathbb{T} \setminus [-1/3, 1/3]$. Let $f \in \mathrm{Hom}(Z, \mathbb{T})$ and color $z \in Z$ by the third that $f(z)$ lies in. If z, z' share a color, then $f(z - z') \in [-1/3, 1/3]$.

Goal: $f(S) \cup f(-S) \subset I$.



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By the Paley-Zygmund inequality $\mathbb{P}(X > 0) \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2} = 1 - o(1)$.

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Fourier analysis and smooth approximation of I shows almost all pairs are good.

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Let X be the number of $f \in \mathcal{F}$ such that $f(S) \subset I_M := \mathbb{Z}_M \setminus \left\{-\left\lceil \frac{M}{3} \right\rceil + 1, \ldots, \left\lceil \frac{M}{3} \right\rceil - 1\right\}$.

Goal: Show $X > 0$ when $|S| = c \log N$ for small c.

First moment: It turns out that $Z[M] = |\mathrm{Hom}(Z, \mathbb{Z}_M)|$. Then

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Counting the number of (f, g) with $d(f, g) = m$ for $m|M$ controls $\mathbb{E}X^2$.

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Thank you!