Chromatic thresholds in Sidon sets

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Sidon sets

Call $S \subset \mathbb{Z}_N$ Sidon if

$$x_1 + x_2 = y_1 + y_2, \quad x_1, x_2, y_1, y_2 \in S$$

implies that $\{x_1, x_2\} = \{y_1, y_2\}$.

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Clearly then

$$|S| = O\left(\sqrt{N}\right).$$

Sidon sets of size $\Omega(\sqrt{N})$ are thus *sparse* and *extremal*.

Can extremality force structure that normally would *not* be present at such sparsity?

Case study: Roth's theorem

There exist subsets of $\mathbb{Z}_N$ of size $\Omega(\sqrt{N})$ with no solution to

$$E : x_1 + x_2 + x_3 + x_4 = 4x_5$$

in distinct variables.

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Theorem (Conlon–Fox–Sudakov–Zhao 2021)

Let N be sufficiently large and $S \subset \mathbb{Z}_N$ be Sidon with $|S| = \Omega(\sqrt{N})$. Then S contains a solution to E with all variables distinct.

Extremality forces solutions to E at a much lower density than normal!

What about other equations?

There exist subsets of $\mathbb{Z}_N$ of size $\Omega(N)$ with no solution to

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Theorem (Liu–Wu–Yang–Zhang 2026+)

Let $A \subset \mathbb{Z}_p$ contain no solutions to E in distinct variables. Then either

$$|A| = o(p) \quad \text{or} \quad \chi\big(\mathrm{Cay}(\mathbb{Z}_p, A)\big) = O(1).$$

All dense counterexamples to a Roth-type theorem for E must generate *highly structured* Cayley graphs.

Sidon analog

Let

$$E : x_1 + x_2 + x_3 + x_4 = 4x_5 + x_6$$

Theorem (Dichotomy, T. 2026+)

Let $S \subset \mathbb{Z}_p$ be Sidon and contain no solutions to E in distinct variables. Then either

$$|S| = o(\sqrt{p}) \quad \text{or} \quad \chi\big(\mathrm{Cay}(\mathbb{Z}_p, S)\big) = O(1).$$

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Theorem (T. 2026+)

For infinitely many N, there exist Sidon $S \subset \mathbb{Z}_N$ free of solutions to E in distinct variables such that

$$|S| = \Omega(\sqrt{N}) \quad \text{and} \quad \chi\big(\mathrm{Cay}(\mathbb{Z}_N, S)\big) = \omega(1).$$

For this E, extremal Sidon sets do *not* behave like dense sets. There exist Sidon obstructions to a Roth-type theorem that also generate unstructured Cayley graphs.

Additive energy

If $\widehat{f}(\xi) := \sum_{n \in \mathbb{Z}_N} f(n) e_N(-\xi n)$, then Parseval gives

$$\mathbb{E}_{\xi \in \mathbb{Z}_N} \left| \widehat{A}(\xi) \right|^2 = \sum_{n \in \mathbb{Z}_N} A(n)^2 = |A| ,$$

i.e. Fourier L^2 norm is just a function of size, and does not "see" any structure.

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The Fourier L^4 norm can be expanded

$$\begin{aligned} \mathbb{E}_{\xi \in \mathbb{Z}_N} \left| \widehat{A}(\xi) \right|^4 = \mathbb{E}_{\xi \in \mathbb{Z}_N} \left| \widehat{A * A}(\xi) \right|^2 &= \sum_{n \in \mathbb{Z}_N} (A * A(n))^2 \\ &= \left| \left\{ (a,b,c,d) \in A^4 : a+b = c+d \right\} \right| . \end{aligned}$$

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This additive energy can generally be as large as $|A|^3$, but for *Sidon* sets is at most $2|A|^2$.

Proof of dichotomy theorem: sketch

Let

$$E : x_1 + x_2 + x_3 + x_4 = 4x_5 + x_6$$

Suppose $S \subset \mathbb{Z}_N$ is Sidon with $|S| = \Omega(\sqrt{N})$.

Goal: Find either

1. A coloring of $\mathrm{Cay}(\mathbb{Z}_N, S)$ with $O(1)$ colors
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The deciding factor will be whether or not $S \cap B$ is empty, where

$$\Gamma := \left\{ \xi \in \mathbb{Z}_N : \left| \widehat{S}(\xi) \right| \text{ large} \right\}, \quad B := \left\{ n \in \mathbb{Z}_N : \| \frac{\gamma n}{N} \|_{\mathbb{T}} \text{ small for each } \gamma \in \Gamma \right\}.$$

Coloring the Cayley graph

Recall: S Sidon,

$$\Gamma := \left\{ \xi \in \mathbb{Z}_N : \left| \widehat{S}(\xi) \right| \geq \nu \sqrt{N} \right\}, \quad B := \left\{ n \in \mathbb{Z}_N : \| \frac{\gamma n}{N} \|_{\mathbb{T}} \leq \rho \text{ for each } \gamma \in \Gamma \right\}.$$

Approach: Decompose $\mathrm{Cay}(\mathbb{Z}_N, S)$ into $O(1)$ independent sets, assuming $S \cap B = \emptyset$.

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Partition $\mathbb{T}^{|\Gamma|}$ into $\lceil 2/\rho \rceil^{|\Gamma|}$ boxes C with side lengths at most $\rho/2$. Color $v \in \mathbb{Z}_N$ by the box containing

$$\phi(v) := \left(\frac{\gamma v}{N} \bmod 1\right)_{\gamma \in \Gamma}.$$

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For v, v' sharing a color, $v - v' \subset B$ because

$$\phi(v - v') = \phi(v) - \phi(v') \subset [-\rho, \rho]^{|\Gamma|}.$$

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Since $S \cap B = \emptyset$, each color class is independent in $\mathrm{Cay}(\mathbb{Z}_N, S)$!

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which is not enough.

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which is not enough. But using the energy bound

$$|\Gamma| \leq \nu^{-4}\mathbb{E}\left|\widehat{S}(\xi)\right|^4/N \leq 2\nu^{-4}|S|^2/N = O(1)$$

Finding a solution to E

Recall: S Sidon, $|S| = \Omega(\sqrt{N})$

$$E : x_1 + x_2 + x_3 + x_4 = 4x_5 + x_6, \quad B := \left\{ n \in \mathbb{Z}_N : \| \frac{\gamma n}{N} \|_{\mathbb{T}} \leq \rho \text{ for each } \gamma \in \Gamma \right\}.$$

Approach: Assume $S \cap B \neq \emptyset$ and fix $y \in S \cap B$. Let M_0, M_y be the number of solutions in S to

$$x_1 + x_2 + x_3 + x_4 = 4x_5 \quad \text{and} \quad x_1 + x_2 + x_3 + x_4 = 4x_5 + y$$

respectively.

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$M_0 \geq \delta N^{3/2}$ for $\delta > 0$.

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Can we line up a solution from M_0 to give one in M_y? $y \in B$ "acts like 0" in Fourier space.

Fourier transfer

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Fourier inversion gives

$$|M_0 - M_y| \leq \frac{1}{N}\sum_{\xi \in \mathbb{Z}_N}\left|\widehat{S}(\xi)^4\widehat{S}(-4\xi)(1 - e_N(y\xi))\right|.$$

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Taking ν small,

$$\sum_{\xi\notin\Gamma}\left|\widehat{S}(\xi)^4\widehat{S}(-4\xi)(1 - e_N(y\xi))\right| \leq 2\nu\sqrt{N}\sum_{\xi\in\mathbb{Z}_N}\left|\widehat{S}(\xi)^3\widehat{S}(-4\xi)\right| \leq \cdots \leq \frac{N}{3}M_0.$$

Open problems

Must every Sidon subset of $\mathbb{Z}_N$ of size $\Omega(\sqrt{N})$ avoiding solutions to

$$2(x_1 - x_2) = 3(x_3 - x_4)$$

generate a cayley graph with bounded chromatic number?

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Thank you!